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STEADY LAMINAR NATURAL CONVECTION PLUMES ABOVE A HORIZONTAL LINE HEAT SOURCE

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Abstract—The extensive past publication concerning the calculation of the behavior of laminar natural convection plumes above energy sources contains numerous elements of uncertainty and confusion concerning proper variables, boundary conditions, and governing equations. This paper, for the first time, states the problem in simplest variables, resolves various apparent redundancies in boundary conditions, and indicates the optimum way to formulate this boundary value problem. Results of numerical calculations are given in terms of the present simple formulation for a wide range of values of Prandtl number, some not having been treated before. Results of experiments with plumes are included to emphasize the various properties of plume flow and to indicate that the large thickness of the boundary region in the range of stable laminar plumes (i.e. at relatively low local Grashof numbers) should encourage the calculation of higher order approximations of the flow.

NOMENCLATURE

c_p, specific heat of the fluid at constant pressure;
d, wire diameter;
f, nondimensional stream function;
g, gravitational acceleration;
Gr, Grashof number defined by equation (6);
k, thermal conductivity of the fluid;
l, wire length;
$\dot{m}$, vertical mass flow rate per unit length;
N, variable as defined by equation (7);
n, exponent defined by equation (7);
Pr, Prandtl number;
Q, heat generated by the line source;
t, fluid temperature;
u, velocity component in x direction;
v, velocity component in y direction;
x, vertical height above the line source;
y, horizontal distance from the mid-plane of the plume;
β, coefficient of volumetric thermal expansion;
ϕ, nondimensional temperature defined by equation (8);
ρ, fluid density;
μ, dynamic viscosity of the fluid;
η, similarity variable defined by equation (4);
ψ, stream function defined by equation (5);
ν, kinematic viscosity of the fluid.

Subscripts

f, Fujii;
0, in the mid-plane;
x, parameter based on distance x;
∞, in the undisturbed fluid.

INTRODUCTION

THIS paper concerns a natural convection plume arising from a horizontal line source of heat in quiescent surroundings of infinite extent. Zeldovich [1] in Russia in 1937 is the first one known to us to have described the natural convection plumes arising from a point and from a horizontal line source of heat. The similarity methods used by Tollmien [2], to solve for the turbulent flow velocity for the 2-dimensional and axisymmetric jet, and by Schlichting [3], to solve for the laminar flow velocities, were employed, and buoyancy and a similarity form of temperature distribution were included. The treatment

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by Zeldovich does not permit a velocity component normal to the plane of symmetry of the plume. However, using the conditions that all the terms of the x momentum equation are of the same order of magnitude and that the heat produced by the source crosses each horizontal plane, expressions are given for the velocity and temperature distributions for both the two dimensional and the radially symmetric cases for both laminar and turbulent flow.

In 1941 Schmidt [4] investigated the behavior of natural convection in a turbulent plume above a line and point source of heat. A similarity technique was used. The governing flow equations were solved by assuming a series solution in terms of the similarity variable. His experimental work included measuring the temperature and velocity above an electrically heated wire.

H. Schuh, in 1948, in the report *Boundary Layers of Temperature* [5], presented a concise analysis of the natural convection boundary layer flow above plane and axially symmetric sources, giving boundary conditions, assuming the form of the similarity variable as originally proposed by Prandtl, and obtaining the coupled differential equations. Schuh refers to an earlier unpublished paper, in which he apparently solved the two point boundary value problem for a plume in a fluid having a $Pr = 0{\cdot}7$. A numerical integration scheme was used, assuming starting values for velocity and temperature at the centerline and correcting these to satisfy imposed conditions at infinity.

A study of natural convection from a point source was reported by Yih [6] in 1951. The coupled equations for axisymmetric laminar flow were solved, analytically, in closed form, for Prandtl numbers of 1 and 2. For the turbulent case he arrived at the temperature and velocity distributions by dimensional analysis coupled with experimental results from a bunsen burner flame. The temperature and velocity distributions were measured by a thermocouple and a small anemometer, respectively. The applicability of the results in the laminar region was at best marginal since the source had a finite size and was also a source of mass.

Yih [7], in 1952, presented a closed form solution for the temperature and velocity distribution for the laminar free convection flow above a line source of heat for Prandtl numbers 2/3 and 7/3.

Measurements of velocity and temperature distributions were performed by Rouse, Yih and Humphreys [8], in 1952, above a line of small gas flames, designed to simulate a line source of heat. Morton, Taylor and Turner [9], in 1956, published a study in which a light fluid was released in a tank of a heavier fluid, with a stable density gradient, to simulate a point source. Morton *et al.* also developed the laminar natural convection theory for maintained and instantaneous sources for plumes in a variable density surrounding medium. This analysis has applicability to the smoke rising from chimneys in a compressible atmosphere.

An experimental study on the weak convective heat transfer from fine heated horizontal wires was performed by Collis and Williams [10] in 1954. The temperature distribution about the wire was determined with an interferometer and the resulting distribution was in partial agreement with Langmuir's stagnant film concept. The study is directed mainly towards hot wire anemometry applications where for a wire on the order of 0·0001 in. dia. and 1 in. long they found that l/d ratios must exceed 20000 for axial conduction through both the wire and gas to be negligible in the integrated effect.

The first work in unsteady natural convection from heated horizontal wires was done in 1956. Ostroumov [11] experimentally investigated the startup phenomena of the plume, comparing the shape and the upward velocity of the "dome" in fluids of various Prandtl numbers.

Mahony [12] in 1956 published an analytic study of natural convection heat transfer at small Grashof numbers from spheres and cylinders to determine the regions in which conduction or convection are the dominant heat transfer mode. It is shown that convection is negligible near the body and becomes as important as conduction at distances from the body on the order of $(Gr)^{-n}$ where n varies from $\frac{1}{3}$ to $\frac{1}{2}$, depending upon body shape. The method used to obtain the range of influence was to match the conduction solution to the convection solution by equating the temperature and the temperature gradient in the vertical direction at some distance above the source.

The problem of laminar natural convection above a linear heat source was again solved analytically by Sevruk [13] in 1958, assuming similarity variables and expressing the solution for the resulting ordinary differential equations in a power series. In 1959 Crane [14] derived the boundary layer equations for a plume above a long thin heated horizontal wire for the case of a gas whose coefficients of viscosity and thermal conductivity vary directly as the absolute temperature using, in effect, the convectional method of variable transformation. A series solution was determined for a particular Prandtl number of 5/9. Another particular Prandtl number case for a line source plume was analyzed by Spalding and Cruddace [15] in 1961 when they simplified and solved the governing differential equations for the natural convection plume in a medium of very high Prandtl number ($Pr = \infty$).

Lee and Emmons [16], in 1961, theoretically and experimentally investigated the behavior of the turbulent natural convection above a line of fire. Theoretically, the governing equations were solved by quadrature for a finite width source, employing the boundary layer assumptions and with the assumptions of lateral entrainment of air and similar Gaussian velocity and temperature profiles at all heights. Experimentally, temperatures were measured with a resistance thermometer and the effect of radiative heat from a luminous flame were determined. The results were in good agreement with theory.

To date, the most thorough treatment of the natural convection plume above a horizontal line and point heat source is the numerical analysis of Fujii [17] in 1963, to which later experimental papers refer for comparison. Fujii solved the two dimensional flow configuration, assuming boundary layer behavior, in closed form for a Prandtl number of 2, and for the axisymmetric case for a $Pr = 1$ and 2. He also used numerical integration to solve the differential equations for Prandtl numbers of 0·01, 0·7 and 10. Because Fujii's work encompasses all of the previous similarity ideas and sets of boundary conditions, the present method of solution along with boundary conditions will be compared with that of Fujii.

The two most recent experimental investigations into the velocity and temperatures profiles around a horizontal wire in air are those by Brodowicz and Kierkus [18] in 1966 and by Forstrom and Sparrow [19] in 1967. Brodowicz and Kierkus used suspended dust particles in air to measure velocities and an interferometer to determine the temperature distribution above a heated wire with an $l/d = 3330$. Their results are only in fair agreement with the plume theory.

Forstrom and Sparrow used a thermocouple to measure the temperature distribution in air at various heat inputs and heights above a wire source. A regular laminar swaying motion of the entire plume was inferred from the regular variation with time of the centerline temperature (at higher heating rates, i.e. higher Grashof numbers). Irregular thermocouple output fluctuations were interpreted as the onset of turbulence. The data and results were interpreted to indicate that a virtual line source should be placed at two wire diameters below the actual wire in order to match the behavior of an actual plume from a wire with the similarity solution from a line source. The value for the temperature similarity variable at the centerline $\phi_f(0)$ determined from measurements was about 15 per cent below the theory results of Fujii.

The most recent work is an analytic study of the laminar free vertical jet, with buoyancy, by Brand and Lahey [20] in 1967. Even though a vertical jet would have mass flow and a vertical component of velocity at the origin, no additional parameters were introduced and the

Table 1. 2-Dimensional and axisymmetric solutions

Author [Ref.]	Configuration	Pr	Method of solution
Schuh [5, unpublished]	2-dimensional	0·7	Numerical integration
	Axisymmetric	0·7	
Yih [6]	Axisymmetric	1, 2	Closed form
Yih [7]	2-dimensional	2/3, 7/3	Closed form
Sevruk [13]	2-dimensional	Variable	Power series solution
Crane [14]	2-dimensional	5/9	Series solution
Spalding and Cruddace [15]	2-dimensional	∞	Approximate closed form
Fujii [17]	2-dimensional	2	Closed form
		0·01, 0·7, 10	Numerical integration
	Axisymmetric	1, 2	Closed form
		0·01, 0·7, 10	Numerical integration
Brand and Lahey [20]	2-dimensional	5/9, 2	Closed form
		0·72, 1, 5, 10	Numerical integration
	Axisymmetric	1, 2	Closed form
		0·72, 5, 10	Numerical integration

formulation of the problem, along with the boundary conditions is, then, identical, but without references to, Fujii's work nor to the extensive literature. In addition to the exact solutions found by Fujii, Brand and Lahey also found closed form solutions for $Pr = 5/9$ for the line source. Their numerical solutions include profiles for Prandtl numbers of 0·72, 5 and 10.

The many and varied studies concerned with the natural convection plume above a horizontal line source of heat prompted this paper, as an attempt to clarify the problem and to bring some order to the diverse formulations in previous studies. The different set of similarity variables used here enables a straightforward analysis of the problem, devoid of the vagueness of many past studies concerning the selection of the appropriate boundary conditions. Additional numerical results are also presented. Methods of solution and cases solved in some of the past studies are summarized in Table 1.

THEORETICAL ANALYSIS

A. *Base flow equations*

The problem of natural convection flow resulting from an infinitely long horizontal line source of heat is considered as a two dimensional laminar, steady state flow. The coordinates and velocity variables are defined in Fig. 1.

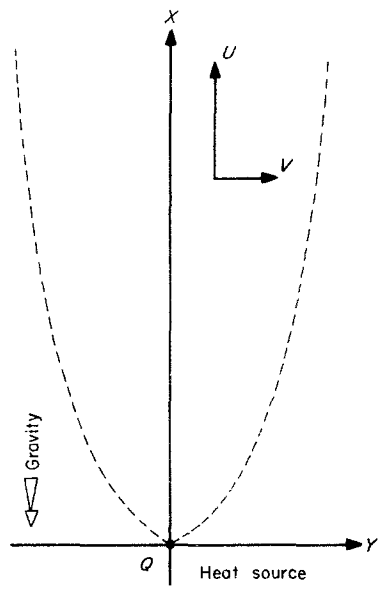

FIG. 1. Geometry.

The governing momentum, energy and continuity equations are simplified by the Boussinesq approximation and by boundary layer assumptions to the following form, in the absence of a stratification in the ambient temperature t_∞.

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g\beta\Delta t + \mu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{k}{\rho c_p}\left(\frac{\partial^2 t}{\partial y^2}\right) \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (3)$$

The viscous dissipation term was not retained in equation (2). Gebhart [21] in 1962 demonstrated that the effect of viscous dissipation in natural convection becomes important only when $g\beta x/c_p = O(1)$, i.e. when the induced kinetic energy becomes appreciable compared to the amount of heat transferred. The quantity $g\beta/c_p$ remains in the range 10^{-7}–10^{-4} for fluids as different as liquid sodium, mercury, gases at ordinary temperature, water and viscous silicones, for the terrestrial level of gravity.

Previous equations are reduced to two ordinary differential equations for certain boundary conditions by introducing a similarity variable $\eta(x, y)$ and a stream function $\psi(x, y)$:

$$\eta = \frac{y}{x}\sqrt[4]{\frac{Gr, x}{4}} \quad (4)$$

$$\psi = 4\nu\sqrt[4]{\left(\frac{Gr, x}{4}\right)}f(\eta), \quad (5)$$

where

$$Gr, x = \frac{g\beta x^3(t_0 - t_\infty)}{\nu^2}. \quad (6)$$

The plume centerline temperature variation with x is initially assumed to be of the power law form Nx^n so that

$$t_0 - t_\infty = Nx^n. \quad (7)$$

It will be shown that this form of the centerline temperature distribution is the appropriate one for certain flow configurations because n may be chosen so that necessary conditions on the heat flow convected thermal energy, across horizontal planes, are satisfied.

The nondimensional temperature excess ratio $\phi(\eta)$ is defined as

$$\phi(\eta) = \frac{t - t_\infty}{t_0 - t_\infty}, \quad (8)$$

so that the local temperature excess is

$$t - t_\infty = Nx^n\phi(\eta). \quad (9)$$

This procedure is in contrast with all earlier work on plumes, in which an undetermined constant arises in the definition of ϕ. The continuity equation (3) is satisfied by ψ and equations (1) and (2) are transformed by (4), (5) and (9) into

$$f''' - (2n + 2)f'^2 + (3 + n)ff'' + \phi = 0 \quad (10)$$

$$\phi'' + Pr[(n + 3)f\phi' - 4nf'\phi] = 0. \quad (11)$$

Up to this point the analysis has been general. The equations apply for boundary layer flow over a vertical plate, for plane sources, and for plumes arising from a horizontal line source. The proper boundary conditions specify the particular case. The boundary conditions along with the values of N and n in equation (7) are determined for a plume by satisfying the necessary conditions on a plume flow resulting from a line source. The energy convected across any horizontal plane (at x) in the plume is

$$Q = \rho c_p\int_{-\infty}^{\infty}(t - t_\infty)u\,dy. \quad (12)$$

In terms of the similarity variables Q becomes

$$Q = \nu 4\rho c_p N\left(\frac{g\beta N}{4\nu^2}\right)^{\frac{1}{4}}x^{3+5n/4}\int_{-\infty}^{\infty}f'(\eta)\,\phi(\eta)\,d\eta \quad (13)$$

Since Q is not a function of x (the only heat addition is due to the line source, there being no other sources) the value of n is found.

$$n = -3/5^* \quad (14)$$

* For a constant flux plate Q must be proportioned to x, so $n = 1/5$, the same value of n applies for a uniform plane source in the mid-plane of a plume. For an isothermal plate, n is equal to 0.

With this value for n, equations (10) and (11) become

$$f''' + \frac{12}{5}ff'' - \frac{4}{5}f'^2 + \phi = 0 \qquad (15)$$

$$\phi'' + \frac{12}{5}Pr(f\phi)' = 0. \qquad (16)$$

These equations for the plume are similar to those obtained by Schuh [5], Fujii [17] and others. A difference arises in the coefficients, because of a different choice of variables.

B. *Boundary conditions*

Establishing boundary conditions for the plume flow equations (15, 16), now that the value of n is determined, is the last step in completely specifying the problem. The governing equations require five independent boundary conditions. The boundary conditions for the line source plume can be generated from physical considerations as follows: The symmetry of the plume with respect to its mid-plane requires that $(\partial t/\partial y)_0 = 0$, $v_0 = 0$, $(\partial u/\partial y)_0 = 0$, $t = t_0$. The symmetry is seen in Fig. 5, each fringe is an isotherm. The above conditions, written in terms of the similarity variables are:

$$\phi'(0) = f''(0) = f(0) = 0 \qquad (17)$$

and

$$\phi(0) = 1. \qquad (18)$$

Energy considerations require that all effects vanish at large values of η, i.e. $u \to 0$ and $t \to t_\infty$. In terms of the dependent functions in this circumstance, we have

$$f'(\infty) \to 0, \qquad \phi(\infty) \to 0. \qquad (19)$$

The problem is apparently over-determined, there are too many boundary conditions. However as will be shown, not all of these conditions are independent.

The energy equation (16) is a perfect differential and may be integrated once to give

$$\phi' + 2{\cdot}4\, Pr\, f\phi = C_1. \qquad (20)$$

The constant C_1 is evaluated from the boundary conditions (17), $\phi'(0)$ and $f(0)$, as zero, and

$$\frac{\phi'}{\phi} = -\, 2{\cdot}4 Prf. \qquad (21)$$

Integrating again can get

$$\phi(\eta) = \phi(0)\, e^{-2{\cdot}4\, Pr \int_0^\eta f\, d\eta} \qquad (22)$$

where $\phi(0) = 1$ from equation (18). Since f is positive and becomes constant for large η, the value of the integral is unbounded and

$$\lim_{\eta \to \infty} \phi(\eta) = 0. \qquad (23)$$

Therefore, the condition $\phi(\infty) \to 0$ is not independent, but is implied by conditions used to evaluate constants of integration. As a result we are free to choose the most convenient set of independent boundary conditions.

The conditions $\phi'(0) = 0$ and $f(0) = 0$ may not be used again since they have already been used to obtain (21). The remaining four independent boundary conditions, two at zero and two at infinity, are sufficient to solve the fourth order system of differential equations given by (15) and (21). However, this would be unwise, in the subsequent numerical solution since two missing conditions must then be guessed at zero in order to start the integration. The problem could also be solved from the integro-differential equation resulting from (15) and (22), with boundary conditions

$$f(0) = 0, f''(0) = 0 \qquad \text{and} \quad f'(\infty) \to 0.$$

Still, two boundary conditions must be satisfied at infinity, and in addition, the integro-differential equation causes difficulties in boundary value problem solution.

However, if one eliminates condition (23), there remain the necessary five conditions for (15) and (16). The problem remains a boundary value problem, but with only one condition to be met at infinity. This procedure represents an appreciable simplification and improvement in method over all previous treatments.

The above results are valid only for the plume arising from a line source and applicable to natural convection flow over a vertical adiabatic surface with a heat source concentrated at the leading edge, since only for $n = -\frac{3}{5}$ is the energy equation an exact differential.

The present boundary conditions and method of solution will be compared with those of previous studies. Since Fujii's [17] work is the most complete and representative of past work, his boundary conditions will be presented and differences from them mentioned. The set of boundary conditions used by Fujii are:

$$f_f(0) = f''_f(0) = \phi'_f(0) = 0 \tag{24}$$

$$f'_f(\infty) = \phi_f(\infty) = 0. \tag{25}$$

The condition $\phi(\infty) = 0$ cannot substitute a condition for the temperature difference along the plume mid-plane. Therefore, Fujii had to introduce the arbitrary normalization:

$$\int_{-\infty}^{\infty} f'_f \phi_f \, \mathrm{d}\eta = 1 \tag{26}$$

[on ϕ, in effect, since $\phi(0)$ is not given], to define the problem. This was an additional condition imposed on the differential equations during the numerical integrations. Fujii stated that the theoretical or numerical calculations cannot be performed without this condition. One would infer that most earlier workers were of the same mind.

Schuh [5], Sevruk [13] and Brand and Lahey [20] use the same five boundary conditions as Fujii. Spalding and Cruddace [15], in solving the flow equations for temperature dependent viscosity, arrive at a set of differential equations similar to Fujii's. They neglected the condition that the normal velocity component is zero at the plume mid-plane but included condition (26), to define the problem.

Equation (26) is the condition of invariant thermal flux in the plume, note equation (13). Similarly, by equating the total momentum in the plume with the total work done by buoyancy, the following relation results.

$$\int_{-\infty}^{\infty} f'^2 \, \mathrm{d}\eta - \tfrac{5}{16} \int_{-\infty}^{\infty} \phi \, \mathrm{d}\eta = 0. \tag{27}$$

Equations (26) and (27) are consequences of the conservation equations and of the boundary conditions peculiar to the plume problem. The latter differentiate the plume flow configuration from that over a heated vertical flat plate, for example.

Since the method of solution presented in this paper does not employ any integral relations, e.g. equation (26) or (27), to perform integration of the governing equations, and since only one condition is lacking at $\eta = 0$ (to start the numerical integration), the method of solution is clear, very simple, and efficient.

C. *Evaluation of temperature, flow rate and velocity distribution*

By defining

$$I = \int_{-\infty}^{\infty} f' \phi \, \mathrm{d}\eta \tag{28}$$

equation (13) can be written as

$$N = \left(\frac{Q^4}{4^3 g\beta\rho^2\mu^2 c_p^4 I^4} \right)^{\frac{1}{5}}. \tag{29}$$

Therefore, equation (7) is completely defined as a function of fluid properties and heat generation rate at the wire and by an integral of the functions f' and ϕ. The local temperature excess is given by (9) as:

$$t - t_\infty = 4^{-\frac{1}{5}} \left[\frac{Q}{Ic_p} \right]^{\frac{4}{5}} \frac{1}{(g\beta\rho^2\mu^2)^{\frac{1}{5}}} x^{-\frac{3}{5}} \phi(\eta) \tag{30}$$

and the mass flow rate in the plume is

$$\dot{m} = 4^{\frac{3}{5}} J \left[\frac{g\beta\rho^2\mu^2 Q x^3}{c_p I} \right]^{\frac{1}{5}} \tag{31}$$

where J is the value of the integral

$$J = \int_{-\infty}^{\infty} f'(\eta) \, \mathrm{d}\eta. \tag{32}$$

Hence we conclude that the maximum temperature in the plume decreases (at constant η) as a minus three-fifths power of the height and that the mass flow rate increases at the same rate. The temperature level increases with four-fifths power of the total heat input and the mass flow increases only to the one-fifth. Viscosity, heat capacity, and density of the fluid have also strong influence on temperature distribution and mass flow rate. The vertical velocity component in the plume is

$$u = 4^{\frac{1}{5}} \left\{\frac{g\beta Q}{c_p I}\right\}^{\frac{2}{5}} \left\{\frac{x}{\mu\rho}\right\}^{\frac{1}{5}} f'(\eta). \tag{33}$$

The horizontal component is

$$v = 4^{\frac{3}{4}} \frac{\nu}{x} Gr^{\frac{1}{4}}, x \left\{\frac{3}{5} f(\eta) - \frac{2}{5} \eta f'(\eta)\right\}. \tag{34}$$

RESULTS AND CONCLUSIONS

Extensive numerical calculations were carried out, in a Prandtl number range from 0·01 to 100, using the more direct formulation and procedure set forth above. The stream function is plotted in Fig. 2. The temperature distribution and vertical velocity component distributions are plotted in Figs. 3 and 4. For plumes, as for convection flows developed over heated vertical plates, the thermal and velocity layers remain coupled in thickness for Prandtl numbers decreasing from 1 to values typical of liquid metals, in contrast to the analogous forced flow case. For increasing Prandtl number, above 1, the thermal layer becomes relatively much thinner than the velocity layer. The decreasing velocity levels in the plume, with increasing Prandtl number, are clearly seen in Fig. 3. Plume symmetry is apparent from the zero slopes at $\eta = 0$ in Figs. 3 and 4. Numerical values of $f'(0)$ and of I and J are given in Table 2. Indications of the thicknesses of the temperature and velocity boundary regions are also given in the table by values of η at $\phi(\eta) = 0{\cdot}01$ and at $u/u_{max} = 0{\cdot}01 = f'(\eta)/f'(0)$.

Temperature and velocity distributions were compared and were in good agreement with those presented by Fujii [17]. The maximum

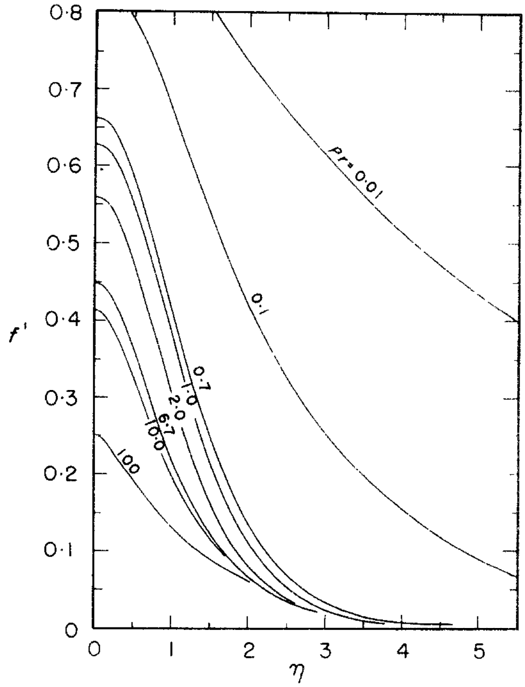

FIG. 3. Computed velocity profiles for a range of Prandtl numbers.

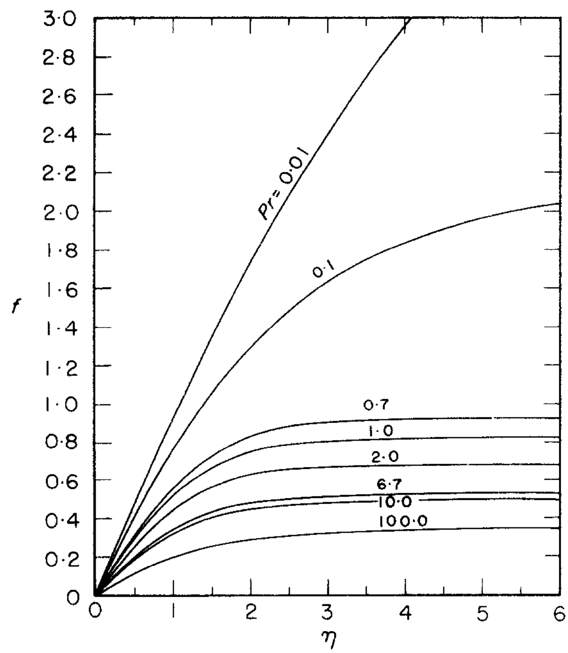

FIG. 2. Computed value of f for a range of Prandtl numbers.

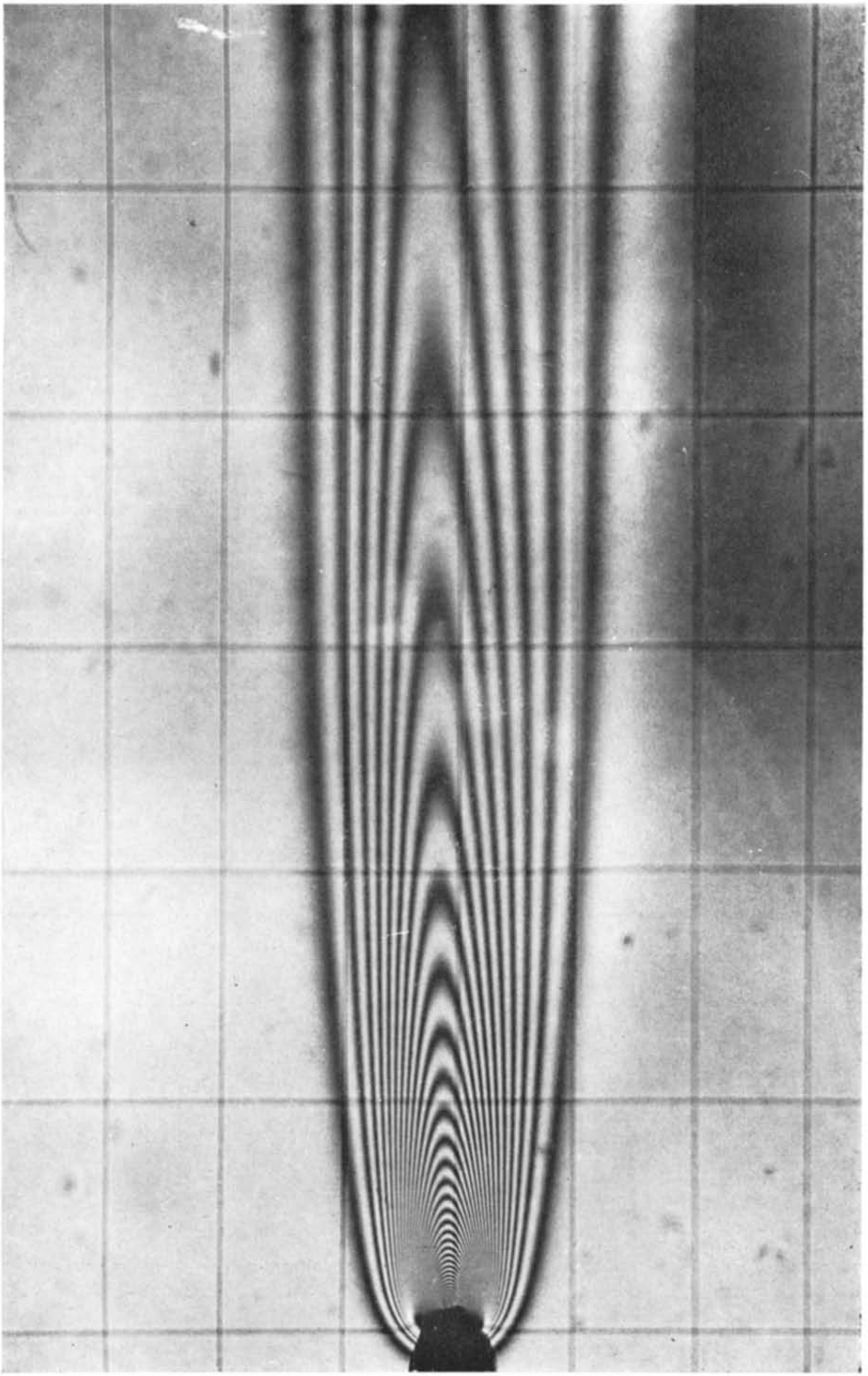

FIG. 5. Interferogram of a plume formed above a heated wire in air.

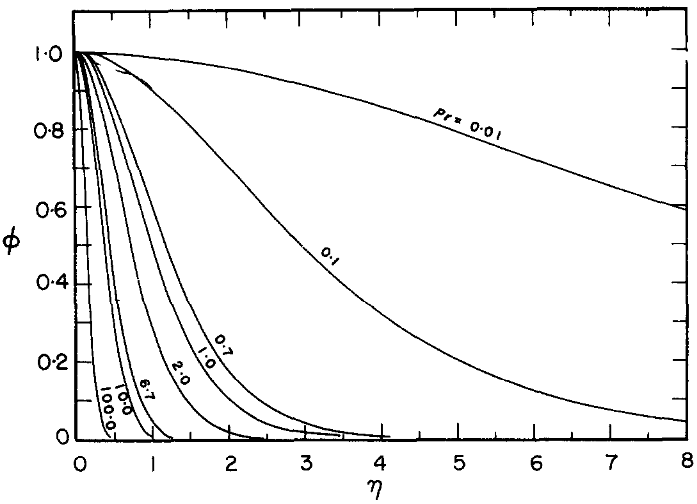

FIG. 4. Computed temperature distribution for a range of Prandtl numbers.

discrepancy in the values of ϕ and f at the mid-plane was about 0·1 per cent.

Figure 5 is an interferogram of a plume in air, at 78°F and 1 atm, formed above a 0·005 in. dia heated wire of 6 in. length at a heating rate of 54 Btu/h ft. The instrument is a 5 in. Mach-Zehnder interferometer with a Mercury vapor source filtered to the green line. The adjustment was to the infinite fringe, each fringe represents an isothermal contour. The constant for these conditions is 7·9°F per fringe. The lens system is conventional, i.e. not anamorphic, the distance scales are the same in both the x- and y-directions. The lines are a wire grid with vertical and horizontal spacing of $\frac{1}{2}$ and $\frac{1}{4}$ in., respectively.

The interferogram indicates clearly the extent of the thermal boundary region of the plume above the source. Since for a Prandtl number of 0·7 the velocity and thermal boundary regions are of almost equal extent, the disturbed region seen is essentially the whole plume.

For this plume the calculated local Grashof number at $x = 2$ in. is $1{\cdot}7 \times 10^6$. The plume half thickness δ, at x, divided by x is about 0·19. The nominal limit of the applicability of boundary layer theory is usually expressed as $(\delta/x) \ll 1$. Now, since laminar plumes are thought, at least by the present writers, to be unstable at even lower local Grashof numbers than laminar flows over surfaces, it is apparent

Table 2. Numerical values of computed parameters

Pr	0·01	0·1	0·7	1·0	2·0	6·7	10·0	100·0
$f'(0)$	0·9751	0·8408	0·6618	0·6265	0·5590	0·4480	0·4139	0·2505
$\phi = 0{\cdot}01$ at $\eta =$	—	11·0	3·9	3·2	2·2	1·2	1·0	0·4
$f'/f'(0) = 0{\cdot}01$ at $\eta =$	14·6	9·3	4·1	3·8	3·7	4·1	4·3	5·6
I	—	3·090	1·245	1·053	0·756	0·407	0·328	—
J	—	4·316	1·896	1·685	1·393	1·094	1·024	—

that higher order approximations should be made in calculating laminar plume flows. We do not at this time know, from experimental observations, the systematic deviations of actual plume flows from the predictions of simplest laminar boundary layer theory. These questions are under study and the first of our experimental investigations will appear in [22].

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PANACHES PERMANENTS ET LAMINAIRES DE CONVECTION NATURELLE AU-DESSUS D'UNE SOURCE DE CHALEUR LINÉAIRE HORIZONTALE

Résumé—Les publications antérieures en nombre considérable concernant le calcul du comportement des panaches de convection naturelle laminaire au-dessus de sources d'énergie, contiennent de nombreux éléments d'incertitude et de confusion sur les variables convenables, les conditions aux limites et les équations qui régissent les phénomènes. Cet article, pour la première fois, pose le problème avec les variables les plus simples, résout les nombreuses redondances dans les conditions aux limites, et indique la façon optimale de formuler ce problème de valeurs aux limites. Les résultats des calculs numériques sont donnés sous la forme de la formulation simple actuelle pour une gamme étendue de valeurs du nombre de Prandtl, certaines n'ayant pas été traitées auparavant. Des résultats d'expériences avec des panaches sont inclus pour mettre en relief les diverses propriétés de l'écoulement du panache et pour indiquer que la grande épaisseur de la région frontière dans la gamme des panaches laminaires stables (c'est-à-dire, à des nombres de Grashof locaux relativement bas) encouragerait des calculs de l'écoulement à des approximations d'ordre plus élevé.

STATIONÄRE, LAMINARE, FREIE KONVEKTION-GEBILDE ÜBER EINER WAAGERECHTEN LININFÖRMIGEN WÄRMEQUELLE

Zusammenfassung—Die zahlreichen Veröffentlichungen über die Berechnung der laminaren Zellströmung, die sich durch freie Konvektion über Energiequellen ausbildet, enthalten viele Unsicherheiten und Unklarheiten über geeignete Variable, Randbedingungen und geltende Gleichungen. In dieser Arbeit wird das Problem zum ersten Mal mit Hilfe einfacher Variabler formuliert; verschiedene überflüssige Randbedingungen werden beseitigt und der beste Weg für die Behandlung dieses Randwertproblems wird gezeigt. Ergebnisse numerischer Rechnungen werden als Ausdrücke einfacher Formulierungen für einen grossen Bereich von Prandtlzahlen, von denen einige früher nicht behandelt wurden, angegeben. Ergebnisse von Experimenten mit solchen Strömungszellen werden ebenfalls angegeben, um ihre unterschiedlichen Eigenschaften hervorzuheben und um zu zeigen, dass die grosse Dicke der Randbezirke im Bereich der stabilen laminaren Konvektionsformen (z.B. bei relativ kleinen lokalen Grashofzahlen) zu einer Berechnung der Strömung mit Näherungen höherer Ordnung ermutigen sollte.

СТАЦИОНАРНАЯ ЛАМИНАРНАЯ СВОБОДНАЯ КОНВЕКЦИЯ НАД ГОРИЗОНТАЛЬНЫМ ЛИНЕЙНЫМ ИСТОЧНИКОМ ТЕПЛА

Аннотация—В имеющихся многочисленных публикациях по расчету ламипарных струек естественной конвекции над источниками энергии содержится много неясного и противоречивого в отношении соответствующих переменных, граничных условий и основных уравнений. В данной статье впервые ставится задача в наиболее простых переменных, разрешаются различные избыточности граничных условий и указывается на оптимальный способ формулировки этой граничной задачи. Результаты численных расчетов выражаются через данную формулировку для широкого диапазона значений критерия Прандтля, некоторые из которых ранее не рассматривались. Результаты зкспериментов со струйками включены с целью подчеркнуть различные свойства струек и указать, уто большая толщина пограничной области в диапазоне стабилъных ламинарных струек (т.е. при относительно низких локальных значениях критерия Грасгофа) должна способствовать проведению расчетов потока с помощью аппроксимаций высшего порядка.